An elementary proof of the uniqueness of the solutions of linear odes

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In this note, we present an elementary proof of the uniqueness of the solutions of the initial value problems of linear ordinary differential equations (odes). This proof uses only elementary results of a first course in Calculus, such as the technique of integration by parts. This approach can be used together with the power series approach (see, e.g., [1]) in order to provide an elementary proof of the existence and uniqueness of the solutions of the initial value problems of linear odes with analytical coefficients. Usually this result is proved only to first-order odes, since the involved technique (the Picard method) is very sophisticated even in this situation. In the higher order case, we would need to apply the Picard method to systems of odes.

Theorem 0.1 *Let y_1 e y_2 be solutions of the linear homogeneous ordinary differential equation*

$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = 0,$$

where a_k is a function of class C^k, satisfying the following initial conditions

$$y(0) = y_0,\ y'(0) = y_1,\ \ldots,\ y^{(n-1)}(0) = y_{n-1}.$$

Then there exists $\varepsilon > 0$ such that $y_1(x) = y_2(x)$, for all x in $(-\varepsilon, \varepsilon)$.

Proof: We prove the theorem in the case $n = 2$, since the general proof is completely analogous.

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Let y_1 and y_2 be solutions of

$$y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0, \tag{1}$$

satisfying the initial conditions $y(0) = y_0$ and $y'(0) = y_1$. Then the function $z = y_1 - y_2$ also satisfies the equation (1) and the following initial conditions $z(0) = z'(0) = 0$. It is sufficient to show that there exists $\varepsilon > 0$ such that $z(x) = 0$, for all x in $(-\varepsilon, \varepsilon)$. Integrating the equation $z'' = -a_1 z' - a_0 z$ from 0 to t, using the Fundamental Theorem of Calculus and that $z'(0) = 0$, we have that

$$z'(t) = -\int_0^t a_1(x)z'(x)dx - \int_0^t a_0(x)z(x)dx.$$

Integrating by part the first integral and using that $z(0) = 0$, it follows that

$$\begin{aligned} z'(t) &= -a_1(t)z(t) + \int_0^t a_1'(x)z(x)dx - \int_0^t a_0(x)z(x)dx \\ &= \int_0^t (a_1'(x) - a_0(x))z(x)dx - a_1(t)z(t). \end{aligned}$$

Integrating this equation from 0 to s, using that $z(0) = 0$ and applying again the Fundamental Theorem of Calculus, we get that

$$z(s) = -\int_0^s \left(\int_0^t (a_1'(x) - a_0(x))z(x)dx - a_1(t)z(t) \right) dt.$$

Fixing $r > 0$, for all $s \in [0, r]$, we have that

$$\begin{aligned} |z(s)| &\leq \int_0^s \left| \int_0^t (a_1'(x) - a_0(x))z(x)dx - |a_1(t)z(t)| \right| dt \\ &\leq \int_0^s \left(\int_0^t |a_1'(x) - a_0(x)||z(x)|dx + |a_1(t)||z(t)| \right) dt \\ &\leq \int_0^s \left(\int_0^t M|z(x)|dx + N|z(t)| \right) dt \\ &= f(s), \end{aligned}$$

where the last equality is just the definition of the function f and

$$M = \max_{x \in [0,r]} |a_1'(x) - a_0(x)| \quad \text{e} \quad N = \max_{x \in [0,r]} |a_0(x)|.$$

Since f is the integral of a nonnegative integrand, it is a nondecreasing function. Thus, for all $\varepsilon \in [0, r]$, it follows that

$$\begin{aligned}
\max_{s\in[0,\varepsilon]} |z(s)| \leq f(\varepsilon) &= \int_0^\varepsilon \left(\int_0^t M|z(x)|dx + N|z(t)| \right) dt \\
&\leq \max_{s\in[0,\varepsilon]} |z(s)| \int_0^\varepsilon \left(\int_0^t M dx + N \right) dt \\
&= \max_{s\in[0,\varepsilon]} |z(s)| \left(M\frac{\varepsilon^2}{2} + N\varepsilon \right),
\end{aligned}$$

Thus we can find $\varepsilon > 0$ such that

$$\max_{s\in[0,\varepsilon]} |z(s)| \leq \frac{1}{2} \max_{s\in[0,\varepsilon]} |z(s)|,$$

implying that $\max_{s\in[0,\varepsilon]} |z(s)| = 0$ and showing that $z(x) = 0$, for all $x \in [0, \varepsilon]$. In an entirely analogous way, we can find $\varepsilon > 0$ such that $z(x) = 0$, for all $x \in [-\varepsilon, 0]$, completing the proof. $\square$

References

[1] E.A. Coddington: An introduction to ordinary differential equations. Prentice-Hall, Englewood Cliffs, N.J. (1961).